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The impact of a plane punch on an elastic half-plane☆

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Abstract

The transient dynamic contact problem of the impact of a plane absolutely rigid punch on an elastic half-plane is considered. The solution of the integral equation of this problem in terms of the unknown Laplace transform of the contact stresses at the punch base is constructed by a special method of successive approximations. The solution of the transient dynamic contact problem is obtained after applying an inverse Laplace transformation to the solution of the integral equation over the whole time range of the impact process, and the law of the penetration of the punch into the elastic medium is determined from a Volterra-type integrodifferential equation. The conditions for the punch to begin to separate from the elastic half-plane are formulated from the solution obtained, and all the stages of the separation process are investigated in detail. The law of the punch motion on the elastic half-plane and the width of the contact area, which varies during the separation, are then determined from the solution of the Volterra-type integrodifferential equation when an additional condition is satisfied.

The zeroth term of the asymptotic form of this problem was obtained previously in Ref. 1–3. An algorithm for constructing the solution was described in Ref. 3, when the boundary conditions of the problem are satisfied at each step of its solution by solving a number of the mixed problems formulated in a special way.

1. Formulation of the problem and its integral equation

We will consider the transient dynamic contact problem of the impact of an absolutely rigid plane punch of width $2a$ and mass m on an elastic half-plane $(-\infty < x < \infty,\ 0 \le y < \infty)$ with initial velocity v_0. The punch is imbedded into the half-plane along the y axis, which is its axis of symmetry. There are no friction and cohesive force between the punch base and the half-plane. At the initial instant of time, the half-plane is at rest, and hence the displacements of the elastic medium $u = u(x, y, t)$, $v = v(x, y, t)$ and their velocities will be zero at $t = 0$. The stresses and displacements are zero at infinity in the medium (when $\sqrt{x^2 + y^2} \to \infty$).

The formulation of this problem in the generally-accepted notation in the theory of elasticity[3–6] includes the following mixed boundary conditions $(t > 0)$:

$$v(x, 0, t) = \varepsilon(t), \quad |x| \le a; \quad \sigma_{yy}(x, 0, t) = 0, \quad a < |x| < \infty; \quad \sigma_{xy}(x, 0, t) = 0, \quad |x| < \infty \tag{1.1}$$

where σ_{yy} and σ_{xy} are the normal and shear stresses and $\varepsilon(t)$ is the law of the punch motion on the elastic half-plane.

Using a Laplace integral transformation (with respect to the time t) with parameter p and a Fourier integral transformation (with respect to the x coordinate), applied in succession to the differential equations of the theory of elasticity[4,5]

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and to the boundary conditions (1.1), taking into account the conditions at infinity and the zero initial conditions, the problem can be reduced to solving an integral equation of the first kind in dimensionless form[1,2]

$$\int_{-1}^{1} \varphi^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = 2\pi f^L(p), \quad |x| \le 1 \tag{1.2}$$

$$k(t) = \int_{\Gamma} K(u) e^{iut} du, \quad K(u) = 2\frac{(1-\beta^2)\sigma_2}{R(u)}, \quad R(u) = (2u^2+1)^2 - 4u^2\sigma_1\sigma_2$$

$$f^L(p) = 2\frac{(1-\beta^2)\mu}{a}\varepsilon^L(p), \quad \beta = \frac{c_2}{c_1}, \quad c_1 = \sqrt{\frac{(\lambda+2\mu)}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}} \tag{1.3}$$

$$\sigma_1 = \sqrt{u^2+1}, \quad \sigma_2 = \sqrt{u^2+\beta^2}, \quad \Lambda = \frac{c_2}{pa}$$

where $\varphi^L(x, p)$ is the Laplace transform of the function $\varphi(x, t)$ – the required distribution function of the contact stresses under the punch, $\varepsilon^L(p)$ is the Laplace transform of the function $\varepsilon(t)$ (1.1), c_1 and c_2 are the propagation velocities of longitudinal and transverse elastic displacement and stress waves, λ and μ are the Lamé coefficients, and ρ is the density of the material of the elastic half-plane. The contour of integration Γ in the complex plane $u = \sigma + i\tau$ extends from $-\infty$ to $+\infty$ along the real axis $\tau = 0$ at an angle of –argp to its positive direction.

2. The zeroth term of the solution of the integral equation and of the problem

The solution of integral Eq. (1.2) in the form of the zeroth term of the asymptotic solution of the integral equation[4] for large p (small Λ) was constructed in Ref. 1,2 in the form of the superposition

$$\varphi_0^L(x, p) = -\varphi_\infty^L\left(\frac{x}{\Lambda}, p\right) + \varphi_{0+}^L\left(\frac{1+x}{\Lambda}, p\right) + \varphi_{0-}^L\left(\frac{1-x}{\Lambda}, p\right), \quad |x| \le 1 \tag{2.1}$$

for $0 \le \Lambda \le 2\beta$. The zero subscript here corresponds to the number of the term of the solution $\varphi^L(x, p)$ of integral Eq. (1.2), and $\varphi_{0\pm}^L(x, p)$ are the solutions of the integral equations

$$\int_{-1}^{\infty} \varphi_{0+}^L\left(\frac{1+\xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = 2\pi f^L(p), \quad -1 \le x < \infty \tag{2.2}$$

$$\int_{-\infty}^{1} \varphi_{0-}^L\left(\frac{1-\xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = 2\pi f^L(p), \quad -\infty < x \le 1 \tag{2.3}$$

while $\varphi_\infty^L(x, p)$ is the solution of the integral equation of the convolution along the whole axis

$$\int_{-\infty}^{\infty} \varphi_\infty^L\left(\frac{\xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = 2\pi f^L(p), \quad -\infty < x < \infty \tag{2.4}$$

Integral Eqs. (2.2) and (2.3), by means of linear replacements, can be reduced to integral equations on the semi-axis, the solution of which is constructed by the Wiener-Hopf method, while the symbol of the kernel $K(u)$ (1.3) is approximated by an expression of special form[1,2]

$$K(u) = \frac{\sqrt{u^2+\beta^2}}{u^2+\eta_R^2}\exp[d_0(\omega(u)+\omega(-u))]$$
$$\omega(u) = (\sqrt{1-iu}-\sqrt{\beta-iu})^2, \quad d_0 = \frac{2}{(1-\sqrt{\beta})^2}\ln\frac{K(0)\eta_R^2}{\beta} \tag{2.5}$$

where $\pm\eta_R$ are Rayleigh poles, determined from the Rayleigh equation $R(u)=0$.

Approximation (2.5) enables us to surmount the main difficulty of the Wiener-Hopf method, namely, factorization of the function $K(u)$, i.e. its representation in the form of the product of two functions $K_\pm(u)$, regular in the upper half-plane (subscript plus) and the lower half-plane (subscript minus). Factorization of the approximation of the function $K(u)$ (2.5) can be achieved by elementary means; we have

$$K_\pm(u) = \frac{\sqrt{\beta\mp iu}}{\eta_R\mp iu}\exp[d_0\omega(\mp u)] \tag{2.6}$$

The solutions of integral Eqs. (2.2) and (2.3), obtained previously in Ref. 1,2, can be represented in the new form

$$\varphi_{0\pm}^L(x,p) = \varphi_\infty^L(x,p)+\tilde{\varphi}_{0\pm}^L(x,p) \tag{2.7}$$

where

$$\tilde{\varphi}_{0\pm}^L(x,p) = -\frac{f^L(p)}{\pi\Lambda K_-(0)}\int_\beta^\infty w(\xi)e^{-\xi x}d\xi, \quad w(\xi) = \frac{1}{\xi}l(\xi)w_*(\xi)$$
$$l(\xi) = \frac{\eta_R-\xi}{\sqrt{\xi-\beta}}, \quad w_*(\xi) = \begin{cases}\exp(d_0(\sqrt{\xi-\beta}-\sqrt{\xi-1})^2), & 1\le\xi<\infty\\ \exp(d_0(1+\beta-2\xi))\cos(2d_0\sqrt{\xi-\beta}\sqrt{1-\xi}), & \beta\le\xi\le 1\end{cases} \tag{2.8}$$

The function $\varphi_\infty^L(x,p)$, which is the solution of integral Eq. (2.4) and obtained in Ref. 1,2 using a Fourier integral transformation, is defined by the formula

$$\varphi_\infty^L(x,p) = \frac{f^L(p)}{\Lambda K(0)} \tag{2.9}$$

The functions $\tilde{\varphi}_{0\pm}^L(x,p)$ (2.8) are Laplace transforms of diffraction waves, generated by the corner points $x=\pm 1$ of the edges of the punch base.

The zeroth term (2.1) of the asymptotic form of the solution of integral Eq. (1.2), taking formula (2.7) into account with the separated Laplace transforms of the diffraction waves, acquires the new form

$$\varphi_0^L(x,p) = \varphi_\infty^L\left(\frac{x}{\Lambda},p\right)+\tilde{\varphi}_{0+}^L\left(\frac{1+x}{\Lambda},p\right)+\tilde{\varphi}_{0-}^L\left(\frac{1-x}{\Lambda},p\right) \tag{2.10}$$

with functions $\tilde{\varphi}_{0\pm}^L(x,p)$ and $\varphi_\infty^L(x,p)$, defined by formulae (2.8) and (2.9) respectively.

After evaluating the inverse Laplace transformation of the function (2.10), we obtain the zeroth term of the solution of the problem in question,[1,2] written in the new form

$$\varphi_0(x,t) = \varphi_\infty\left(\frac{ax}{c_2},t\right)+\tilde{\varphi}_{0+}\left(\frac{a(1+x)}{c_2},t\right)+\tilde{\varphi}_{0-}\left(\frac{a(1-x)}{c_2},t\right), \quad 0<t<t_1; \quad t_1 = \frac{2a}{c_1} \tag{2.11}$$

where

$$\varphi_\infty(u,t) = \frac{a}{c_2K(0)}H(t)(\dot{f}(t)+f(0)\delta(t)) \tag{2.12}$$

$$\tilde{\varphi}_{0\pm}(u,t) = -\frac{1}{\pi K_-(0)}\frac{a}{c_2}\frac{H(t-\beta u)}{\sqrt{u}}\int_0^{t-\beta u}\dot{f}(\tau)q(t-\tau,u)w_*\left(\frac{t-\tau}{u}\right)d\tau$$
$$f(t) = 2\frac{(1-\beta^2)\mu}{a}\varepsilon(t), \quad q(t,u) = \frac{u\eta_R - t}{t\sqrt{t-\beta u}} \tag{2.13}$$

$H(t)$ is the Heaviside function, and a dot denotes a derivative of the function $f(t)$ with respect to time t.

The functions $\tilde{\varphi}^L_{0\pm}(x, t)$ (2.13) describe diffraction waves, generated by the corner points of the edges of the punch base $x=\pm1$ at the initial instant of time ($t=0$) when the punch is imbedded. Here we are considering the front of the longitudinal stress (see the expression for $q(t, u)$) with the root singularity $(c_l t - a(1 \pm x))^{-1/2}$ in the case $f(0) \neq 0$ or smooth front of the type $(c_l t - a(1 \pm x))^{1/2}$ in the case when $f(0)=0$, and the front of the transverse wave in $w_*(t)$, a constant singularity of the root type $(1 \pm x)^{-1/2}$ (independent of time during the period the punch is imbedded and up to the instant when it begins to separate from the elastic medium) at the corner points of the edges of the punch base $x=\pm1$.

The solution (2.11)–(2.13) obtained is the zeroth term of the solution of the problem and is defined for the time interval $0<t<t_1$, i.e. until the diffraction waves (2.13), generated by the corner points of the edges of the punch base, reach the corner points of the edges of the punch base opposite to them.

The law of the punch motion $\varepsilon(t)$, like the motion of its centre of mass, is found from the differential equation[1,2]

$$m\ddot{\varepsilon}(t) = Q(t), \quad \varepsilon(0) = \varepsilon_0, \quad \dot{\varepsilon}(0) = \upsilon_0 \tag{2.14}$$

for the period of time $0<t<t_1$, during which the force $Q(t)$ of the elastic resistance of the medium to the penetration of the punch is equal to the force of contact action of the punch on the elastic medium, taken with the opposite sign

$$P(t) = a\int_{-1}^{1}\varphi(x,t)dx \tag{2.15}$$

In the time interval considered, by formula (2.11), we have

$$P(t) = a\int_{-1}^{1}\varphi_0(x,t)dx = P_\infty(t) + \tilde{P}_{0+}(t) + \tilde{P}_{0-}(t), \quad 0<t<t_1 \tag{2.16}$$

$$P_\infty(t) = a\int_{-1}^{1}\varphi_\infty\left(\frac{ax}{c_2}, t\right)dx = \frac{at_2}{K(0)}(\dot{f}(t) + f(0)\delta(t))H(t), \quad t_2 = \frac{2a}{c_2} \tag{2.17}$$

$$\tilde{P}_{0\pm}(t) = a\int_{-1}^{1}\tilde{\varphi}_{0\pm}\left(\frac{a(1\pm x)}{c_2}, t\right)dx = -\frac{a}{\pi K_-(0)}H(t)f(t)\int_\beta^\infty\frac{w(\xi)}{\xi}d\xi \tag{2.18}$$

The formulae obtained enable us to calculate the change in the scalar field of the contact stresses under the punch $\varphi_0(x, t)$ and to determine the law of the punch penetration $\varepsilon(t)$, the penetration rate $\varepsilon(t)$ and other characteristics of the process of the punch penetration during the period of time $0<t<t_1$, provided that the punch does not separate from the elastic medium during this period.

3. Separation of the punch from the elastic medium

In a number of papers (Ref. 1–3,6 etc.), in which analytical methods were used when solving transient dynamic contact problems, particular attention was devoted to investigating the process of the penetration of a plane rigid punch into an elastic medium, assuming that either the law of the punch penetration is known or it is known that the punch adheres to the surface of the elastic medium, while the width of the contact area is fixed and is identical with the width of the punch. An investigation of the processes accompanying impact presupposes an investigation not only of the process by which the punch penetrates into the elastic medium, but also its second phase, namely, the extrusion of

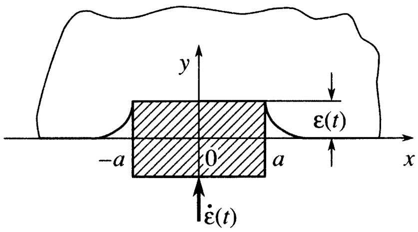

Fig. 1.

the punch, which includes the separation of the punch from the elastic medium. An analysis of the separation of the punch enables additional important characteristics of the processes accompanying the impact to be established and investigated, such as the duration of the contact, the rate of detachment, etc.

For a fixed width of the contact area $2a$, which is identical with the width of the punch, when a solution of problem (2.11)–(2.13) is available for $0 < t < t_1$, it remains to determine the law of the punch penetration $\varepsilon(t)$ from Eq. (2.14) and to substitute it into relations (2.11)–(2.13). Formulae (2.11)–(2.13) show that during the penetration of a plane punch the corner points of the edges of the punch base $x = \pm 1$ are stress concentration points, and fracture of the surface of the elastic medium outside the punch base occurs (Fig. 1). The direction of the velocity of the punch motion $\varepsilon(t)$ in the figures presented here is denoted by an arrow; the length of the arrow corresponds to the value of the velocity. The elastic resistance force of the medium to the punch penetration $Q(t)$ $(0 < t < t_1)$ initially stops the punch (in the deepened position), and then begins to press it out, which is accompanied by local bulging of the medium under the punch. As a result, an instant of time $t = t_*$ arrives when the corner points at the edges of the punch base separate from the elastic medium and lose their fundamental property, namely, the property of stress concentrators, and the fracture on the surface outside the punch then disappears and the surface becomes smooth (Fig. 2). From a mathematical point of view this means that, in the solution $\varphi_0(x, t)$ of problem (2.11)–(2.13) $t = t_*$, the constant (time-independent) root-type singularity $\omega_0(x, t)(1 - x^2)^{-1/2}$ at the corner points of the edges of the punch base disappears. This can only occur if the coefficient

$$C_0(t, a) = \lim_{x \to \pm 1} \sqrt{1 - x^2}\varphi_0(x, t) = \lim_{x \to \mp 1 \pm 0} \sqrt{1 \pm x}\varphi_{0\pm}\left(\frac{a(1 \pm x)}{c_2}, t\right) \tag{3.1}$$

which occurs with this singularity, vanishes: $C_0(t_*, a) = 0$, $0 < t_* < t_1$. Then, to determine the instant of time $t = t_*$ when the punch begins to detach it is necessary to solve the equation

$$C_0(t, a) = -\frac{H(t)}{\pi K_-(0)}\sqrt{\frac{a}{c_2}}\int_0^t \frac{\dot{f}(\tau)}{\sqrt{t - \tau}}d\tau = 0, \quad 0 < t < t_1 \tag{3.2}$$

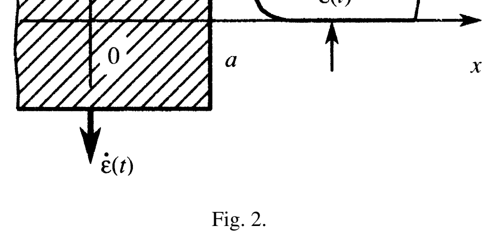

Fig. 2.

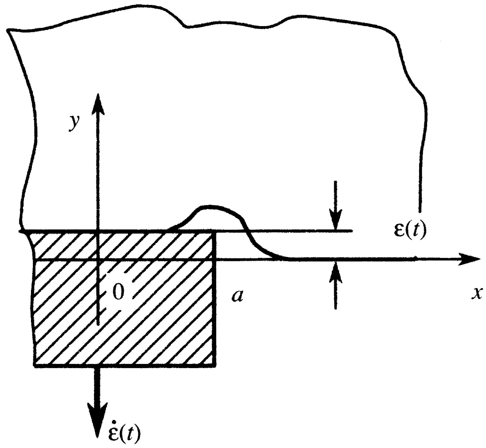

Fig. 3.

with $a = \text{const}$. Then, at the instant of time $t = t_*$ the solution $\varphi_0(x, t)$ (2.11)–(2.13) of integral Eq. (1.2) changes from the class of integrable functions $\varphi_0(x,t) = \omega_0(x,t)(1-x^2)^{-1/2}$($\omega_0(x, t) \in C_{[-1,1]}$) for each $t \in (0, t)$ in the section $|x| < 1$) which allow of a root singularity at the corner points of the edges of the punch base, into a class of continuous functions $\varphi_0(x, t) \in C_{[-1,1]}$) in the section $|x| \le 1$.

After the corner points of the edges of the punch base $x = \pm 1$ separated from the elastic medium when $t > t_*$ ($t_* < t_1$) the half-width of the contact area a begins to change with time (Fig. 3). Since the corner points of the edges of the punch base when $t = t_*$ separated from the elastic medium, the solution $\varphi_0(x, t)$ (2.11)–(2.13) when $t \ge t_*$ should be determined in the class of continuous functions, which can only be obtained by satisfying condition (3.2) when $t > t_*$. This can be achieved only by choosing the half-width of the contact area a, as a result of which, for each $t > t_*$, condition (3.2) is converted into an algebraic equation for determining the value of the function $a(t)$ corresponding to the instant of time t

$$C_0(t, a(t)) = 0, \quad t_* < t < t_1 \tag{3.3}$$

It follows from formula (3.2) that $C_0(t, a(t))$ depends on the law of the punch penetration $\varepsilon(t)$, and hence Eq. (3.3) is solved for $a(t)$ at each step with respect to t of the numerical integration of the differential equation of the punch motion (2.14), the right-hand side of which depends in turn not only on t but also on the half-width of the contact area $a(t)$, which occurs in formulae (2.15)–(2.18). It must be emphasised that in relation (3.3) a does not depend on the integration variable τ and depends solely on t.

It can be shown that the solution $a(t)$ of Eq. (3.3) is a decreasing function for $t_* < t < t_{**} < t_1$, where t_{**} is the time the punch separates from the elastic medium, i.e. the time when $a(t_{**}) = 0$. The function $P(t)$ (2.16)–(2.18), which defines the right-hand side of differential Eq. (2.14), is a function which decreases as $a \to 0$, for which the following estimate holds

$$P(t) = k_0(t)a + O(a\sqrt{a}) \quad \text{as} \quad a \to 0, \quad t_* < t < t_{**} \tag{3.4}$$

An important characteristic of the separation of the punch from the elastic medium is its rate of separation $v_* = \dot{\varepsilon}(t_{**})$, determined in the course of solving problem (2.14) simultaneously with condition (3.3); the rate of change of the half-width of the contact area $\dot{a}(t)(t_* < t < t_{**})$ is then also determined.

It should be noted that the expression for $C_0(t, a)$ and the integral in (3.2) is evaluated by parts. We then obtain the condition

$$C_0(t, a(t)) = H(t)\left(\int_0^t P(\tau)\sqrt{t-\tau}d\tau + mv_0\sqrt{t}\right), \quad 0 < t < t_1 \tag{3.5}$$

in which $P(\tau)$, like $\dot{f}(\tau)$, depends on $a(t)$ (see formulae (2.16)–(2.18)), and m is the mass of the punch. The use of condition (3.5) is often more convenient than (3.2) when solving the problem numerically.

4. The solution of the problem over the whole period of time of the impact

We will construct a solution of the problem for the time interval $t_1 < t < 2t_1$, assuming that separation of the punch from the elastic medium does not occur during the time $t \in (0, t_1)$. Diffraction waves $\varphi_0(x, t)$, generated by the edges of the punch base at the initial instant of impact, travel along the whole punch base from one edge to the other and, when $t = t_1$, generate, at opposite corner points of the edges of the punch base $x = \pm 1$, new diffraction waves $\tilde{\varphi}_{1\pm}(x, t)$, which reach the free surface of the elastic medium outside the punch $-\infty < x < -1$, $1 < x < \infty$, as a result of which the formulation of the initial problem breaks down for $t > t_1$. To remove the load from the diffraction wave $\tilde{\varphi}_{0\pm}(x, t)$ on the surface outside the punch when $t_1 < t < 2t_1$, where a stress-free surface should be, it is necessary to correct the boundary conditions of the initial problem on the surface outside the punch, as was done earlier in Ref. 3.

To obtain the first term $\varphi_1(x, t)$ of the solution of the problem in the new time interval $t_1 < t < 2t_1$, one must determine $\varphi_1^L(x, p)$ for $2\beta < \Lambda < 4\beta$, which reduces to constructing the first step of a special method of successive approximations of the solution of integral Eq. (1.2). With this aim in view, we subtract and add $\varphi_0^L(x, p)$ (2.1) to $\varphi^L(x, p)$ in integral Eq. (1.2) and, taking relations (2.2)–(2.4) for $\varphi_{0\pm}^L((1 \pm x)/\Lambda, p)$, $\varphi_\infty^L(x/\Lambda, p)$ into account, and also the structure of $\varphi_{0\pm}^L((1 \pm x)/\Lambda, p)$ (2.7), we obtain the new integral relation

$$\int_{-1}^{1} \varphi_1^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = H(1 + x) \int_{1}^{\infty} \tilde{\varphi}_{0+}^L\left(\frac{1 + \xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi + \\ + H(1 - x) \int_{-\infty}^{1} \tilde{\varphi}_{0-}^L\left(\frac{1 - \xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi \tag{4.1}$$

The subscript unity denotes the first term of the solution of integral Eq. (1.2)

$$\varphi_1^L(x, p) = \varphi^L(x, p) - \varphi_0^L(x, p) \tag{4.2}$$

The diffraction waves $\tilde{\varphi}_{0\pm}^L(x, P)$ for $2\beta < \Lambda < 4\beta$ in integral relation (4.1), corresponding to the time interval $t_1 < t < 2t_1$, exist outside the punch base. In order to avoid the emergence of diffraction waves, it is necessary, on the right-hand side of relation (4.1), to reflect the load specularly with respect to the corner points of the punch base $x = \pm 1$ in the x axis, which is easily done if we interchange the Heaviside functions $H(1 \pm x)$, in front of the integrals on the right-hand side of Eq. (4.1). This yields the integral equation for determining $\varphi_1^L(x, p)$, the solution of which can be constructed in the form of the superposition of new diffraction waves

$$\varphi_1^L(x, p) = \tilde{\varphi}_{1+}^L\left(\frac{1 + x}{\Lambda}, p\right) + \tilde{\varphi}_{1-}^L\left(\frac{1 - x}{\Lambda}, p\right) \tag{4.3}$$

The functions on the right-hand side of Eq. (4.3) are solutions of the following integral equations on the semiaxes

$$\int_{-1}^{\infty} \tilde{\varphi}_{1+}^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = \int_{-\infty}^{-1} \tilde{\varphi}_{0-}^L\left(\frac{1 - \xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi, \quad -1 \le x < \infty \tag{4.4}$$

$$\int_{-\infty}^{1} \tilde{\varphi}_{1-}^L(\xi, p) k\left(\frac{\xi - x}{\Lambda}\right) d\xi = \int_{1}^{\infty} \tilde{\varphi}_{0+}^L\left(\frac{1 + \xi}{\Lambda}, p\right) k\left(\frac{\xi - x}{\Lambda}\right) d\xi, \quad -\infty < x \le 1 \tag{4.5}$$

The solutions of these equations, like integral Eqs. (2.2) and (2.3), can be constructed by the Wiener-Hopf method,[7] where it is required to invert the same integral operator as at the zeroth step of the method of successive approximations when inverting the integral operators (2.2) and (2.3). Making the linear replacements of variables in integral Eqs. (4.4) and (4.5)

$$x = \mp 1 \pm \Lambda x', \quad \xi = \mp 1 \pm \Lambda \xi'$$

and omitting the primes, we obtain the Wiener-Hopf integral equation on the semiaxis

$$\int_0^\infty \tilde{\varphi}_{1\pm}^L(\xi, p)k(\xi - x)d\xi = \int_0^\infty \tilde{\varphi}_{0\mp}^L\left(\frac{2}{\Lambda} - \xi, p\right)k(\xi - x)d\xi, \quad 0 \le x < \infty \tag{4.6}$$

with respect to the unknown Laplace transforms $\varphi_{1\pm}^L(x, p)$ with the same kernel $k(t)$ (1.3) as integral Eq. (1.2). In view of the evenness of the problem $\tilde{\varphi}_{1+}^L(x, p) = \tilde{\varphi}_{1-}^L(x, p)$, when solving integral Eqs. (2.2) and (2.3) it is sufficient, as previously, to solve one of the integral equations of (4.6).

Using the standard Wiener-Hopf method to solve integral Eq. (4.6), in doing which one can use a special approximation of the symbol of the kernel of integral Eq. (1.2) $K(u)$ of the form (2.5), we obtain the following solution of integral Eq. (4.6)

$$\tilde{\varphi}_{1\pm}^L(x, p) = -\frac{1}{\pi^2}\int_{-\infty}^0 \tilde{\varphi}_{0\pm}^L\left(\frac{2}{\Lambda} - \xi, p\right)d\xi\int_\beta^\infty \eta_1 w(\eta_1)e^{-\eta_1 x}d\eta_1\int_\beta^\infty \frac{r(u_1)}{\eta_1 - u_1}e^{u_1\xi}du_1, \quad r(u) = \frac{w_*(u)}{l(u)} \tag{4.7}$$

The inner integral in Eq. (4.7) is understood in the sense of the Cauchy principal value[7] and contains two points of discontinuity of the integrand $u_1 = \eta_R$, $u_1 = \eta_1$ in the semi-infinite interval of integration.

Substituting expressions (2.8) into Eq. (4.7) and evaluating the quadratures thereby obtained, we obtain solutions of integral Eq. (4.6) in the expanded form

$$\begin{aligned}\tilde{\varphi}_{1\pm}^L(x, p) &= \frac{1}{\pi^3 K_-(0)}\frac{f^L(p)}{\Lambda}\int_\beta^\infty w(\eta_0)d\eta_0\int_\beta^\infty \eta_1 w(\eta_1)\times\\ &\times \exp\left(-\eta_0\frac{2}{\Lambda} - \eta_1 x\right)d\eta_1\int_\beta^\infty \frac{r(u_1)du_1}{(\eta_0 + u_1)(\eta_1 - u_1)}\end{aligned} \tag{4.8}$$

Reverting to the old variables, we obtain the required solution in the form of superposition (4.3).

Hence, the solution of integral Eq. (1.2), determined by the special method of successive approximations, taking formulae (4.2), (2.11) and (4.3) into account, acquires the following form for the interval $2\beta < \Lambda < 4\beta$

$$\varphi^L(x, p) = \varphi_\infty^L\left(\frac{x}{\Lambda}, p\right) + \sum_{k=0}^{1}\left(\tilde{\varphi}_{k+}^L\left(\frac{1 + x}{\Lambda}, p\right) + \tilde{\varphi}_{k-}^L\left(\frac{1 - x}{\Lambda}, p\right)\right) \tag{4.9}$$

The functions $\tilde{\varphi}_{0\pm}^L(x, p)$, $\tilde{\varphi}_{\pm}^L(x, p)$ and $\varphi_\infty^L(x, p)$ are defined by relations (2.8), (4.8) and (2.9) respectively. It should be noted that the function $\varphi^L(x, p)$ is defined in the class of functions with an integrable singularity at the corner points $x = \pm 1$ of the edges of the base of a plane punch, i.e.

$$\varphi^L(x, p) \in \omega_*(x, p)(1 - x^2)^{-1/2}, \quad \omega_*(x, p) \in C_{[-1, 1]}. \tag{4.10}$$

On the other hand, relation (4.1) is also a solution of the problem in Laplace transforms for $2\beta < \Lambda < 4\beta$. Then, changing in relation (4.9) to the Laplace originals, we obtain a solution of the problem for the time interval $0 < t < 2t_1$

$$\varphi(x, t) = \varphi_\infty\left(\frac{ax}{c_2}, t\right) + \sum_{k=0}^{1} H((k + 1)t_1 - t)\left(\tilde{\varphi}_{k+}\left(\frac{a(1 + x)}{c_2}, t\right) + \tilde{\varphi}_{k-}\left(\frac{a(1 - x)}{c_2}, t\right)\right) \tag{4.11}$$

where

$$\tilde{\varphi}_{1\pm}(u,t) = \frac{H(t-t_1-u\beta)}{2\pi^3 K_-(0)\sqrt{u}} \int_0^{t-t_1-u\beta} \dot{f}(\tau)d\tau \int_\beta^{(t-\tau-u\beta)/t_2} w(\eta_0)q_*\left(\frac{t-\tau}{t_2}-\eta_0, u\right)\times$$
$$\times w_*\left(\frac{t-\tau-\eta_0 t_2}{u}\right)d\eta_0 \int_\beta^\infty \frac{r(u_1)du_1}{(\eta_0+u_1)((t-\tau)/t_2-\eta_0-uu_1t_2^{-1})}, \quad q_*(t,u) = \frac{u\eta_R t_2^{-1}-t}{\sqrt{t-u\beta t_2^{-1}}} \tag{4.12}$$

while the functions $\tilde{\varphi}_{0\pm}((a(1+x))/(c_2), t)$ and $\varphi_\infty(ax)/(c_2), t$ are defined by formulae (2.12) and (2.13).

The solution (4.12) for $t_1 < t < 2t_1$ is constructed in the class of functions similar to the class (4.10).

To determine the law of the punch motion $\varepsilon(t)$ from the solution of problem (2.14) in the new time inteval $t_1 < t < 2t_1$, we first find the function $Q(t) = -P(t)$ using formulae (2.15) and (4.11). We have

$$P(t) = P_\infty(t) + \sum_{k=0}^{1} H((k+1)t_1 - t)(\tilde{P}_{k+}(t) + \tilde{P}_{k-}(t)), \quad 0 < t < 2t_1 \tag{4.13}$$

where

$$\tilde{P}_{1\pm}(t) = \frac{aH(t-t_1)}{\pi^3 K_-(0)} \int_\beta^{t/t_2} w(\eta_0)f(t-t_2\eta_0)d\eta_0 \int_\beta^{t/t_2-\eta_0} w(\eta_1)d\eta_1 \int_\beta^0 \frac{r(u_1)du_1}{(\eta_0+u_1)(\eta_1-u_1)} \tag{4.14}$$

while the functions $P_\infty(t)$ and $\tilde{P}_{0\pm}(t)$ are defined by formulae (2.17) and (2.18).

The condition for the punch to start to separate from the elastic half-plane in the time interval $t_1 < t < 2t_1$ can be obtained using the same procedure as in the previous time interval $0 < t < t_1$ by equating the coefficient $C_1(t, a)$ to zero for a constant singularity $(1-x^2)^{-1/2}$ at the corner points of the edges of the base of a plane punch (with $a = \text{const}$), similar to condition (3.1),

$$C_1(t,a) = \lim_{x\to\pm1}\sqrt{1-x^2}\varphi(x,t) = \lim_{x\to\mp1\pm0}\sqrt{1\pm x}\sum_{k=0}^{1} H((k+1)t_1-t)\tilde{\varphi}_{k\pm}\left(\frac{a(1\pm x)}{c_2}, t\right) \tag{4.15}$$

if condition (3.2) is not satisfied in the previous time interval $0 < t < t_1$. Taking formula (4.11) into account, we conclude that the achievement of the condition for the punch to start to detach in the time interval $t_1 < t < 2t_1$ leads to the following relation

$$\sum_{k=0}^{1} H((k+1)t_1 - t)C_k(t,a) = 0 \tag{4.16}$$

where

$$C_1(t,a) = \frac{H(t-t_1)}{\sqrt{2}\pi^3 K_-(0)}\sqrt{\frac{a}{c_2}} \int_0^{t-t_1} \dot{f}(\tau)d\tau \int_\beta^{(t-\tau)/t_2} \frac{w(\eta_0)d\eta_0}{\sqrt{(t-\tau)/t_2-\eta_0}} \int_\beta^\infty \frac{r(u_1)}{(\eta_0+u_1)}du_1 \tag{4.17}$$

while the function $C_0(t, a)$ is given by formula (3.2).

From Eq. (4.16) we can determine the instant of time $t = t_*$ when the punch begins to separate from the elastic half-plane in the time interval $t_1 < t < 2t_1$ when $a = \text{const}$. In this time interval solution (4.11) when $t = t_*$ converts from the class of integrable functions into the class of continuous functions, and when $t > t_*$ condition (4.16) is converted into an equation from which we find the variable half-width of the contact area $a(t)$.

Condition (4.16), like (3.3), depends on the law of the punch penetration $\varepsilon(t)$, determined from the solution of problem (2.14), which contains an integrodifferential equation, where $Q(t) = -P(t)$ is found from the formula for $P(t)$ (4.13) in the interval $t_1 < t < 2t_1$. Then, for the functions $a(t)$ and $\varepsilon(t)$ in the interval $t_* < t < 2t_1$ it is necessary, as in Section 3, at each step of the numerical solution of the integrodifferential equation, to determine from Eq. (4.16) the half-width of the contact area $a(t)$ corresponding to this instant of time.

The algorithm for solving the problem at the zeroth step ($0<t<t_1$) and the first step ($t_1<t<2t_1$), if up to the time $t<2t_1$ the punch has not separated from the elastic medium, can be extended to obtain the solution of the problem at the n-th step of the solution for the time interval

$$nt_1 < t < (n+1)t_1, \quad n = 0, 1, 2, \dots \tag{4.18}$$

of the impact of a plane punch on an elastic half-plane.

The scalar field of the contact stresses in the contact area for $nt_1<t<(n+1)t_1$ is calculated from the formula

$$\varphi(x, t) = \varphi_\infty\left(\frac{ax}{c_2}, t\right) + \sum_{k=0}^{n} H((k+1)t_1 - t)\left(\tilde{\varphi}_{k+}\left(\frac{a(1+x)}{c_2}, t\right) + \tilde{\varphi}_{k-}\left(\frac{a(1-x)}{c_2}, t\right)\right) \tag{4.19}$$

Here

$$\tilde{\varphi}_{k\pm}(u, t) = \frac{(-1)^{k+1}H(t-kt_1-u\beta)}{2\pi^{2k+1}K_-(0)}\sqrt{\frac{t_2}{u}}\int_0^{t-kt_1-u\beta} f(\tau)R_k\left(\frac{t-\tau}{t_2}, u\right)d\tau, \quad k = 2, 3, \dots, n$$

$$R_k(t, u) = \int_\beta^{t-(k-1)\beta-u\beta t_2^{-1}} w(\eta_0)F_k(\eta_0, t, u)d\eta_0$$

$$F_k(\eta_0, t, u) = \int_\beta^{\theta_0}\int_\beta^{\theta_1}\dots\int_\beta^{\theta_{k-3}}\prod_{i=1}^{k-2}\eta_i w(\eta_i)d\eta_i\int_0^{\theta_{k-2}}\eta_{k-1}w(\eta_{k-1})q_*(t-\varsigma_{k-1}, u)w_*\left(\frac{t-\varsigma_{k-1}}{ut_2^{-1}}\right)\times$$

$$\times G_k(\eta_0, \eta_1, \dots, \eta_{k-1}, t, u)d\eta_{k-1}$$

$$G_k(\eta_0, \eta_1, \dots, \eta_{k-1}, t, u) = \int_\beta^\infty\frac{r(u_k)G^*_{k-1}(\eta_0, \eta_1, \dots, \eta_{k-1})du_k}{(\eta_{k-1}+u_k)(t-\varsigma_{k-1}-uu_k\beta t_2^{-1})} \tag{4.20}$$

$$G^*_k(\eta_0, \eta_1, \dots, \eta_k) = \int_\beta^\infty\int_\beta^\infty\dots\int_\beta^\infty\prod_{i=1}^{k}\Omega_i(u_i, \eta_i, \eta_{i-1})du_i, \quad \Omega_i(a, b, c) = \frac{r(a)}{(c-a)(b-a)}$$

$$\theta_i = \gamma_i - \frac{u\beta}{t_2}; \ \gamma_i = \frac{t}{t_2} - \beta(k-2-i)H(k-2-i) - \varsigma_i, \quad \varsigma_i = \sum_{j=0}^{i}\eta_j$$

All the integrals over u_k are understood in the sense of the Cauchy principal value. The functions $\varphi_\infty(u, t)$ are found from formulae (2.12), (2.13) and (4.12).

The contact force of the punch on the elastic medium is given by the formula

$$P(t) = P_\infty(t) + \sum_{k=0}^{n} H((k+1)t_1 - t)(\tilde{P}_{k+}(t) + \tilde{P}_{k-}(t)), \quad nt_1 < t < (n+1)t_1$$

$$\tilde{P}_{k\pm}(t) = \frac{(-1)^{k+1}aH(t-kt_1)}{\pi^{2k+1}K_-(0)}\int_\beta^{t/t_2-k\beta} w(\eta_0)\Phi_k(\eta_0, t)d\eta_0, \quad k = 2, 3, \dots, n \tag{4.21}$$

$$\Phi_k(\eta_0, t) = \int_\beta^{\gamma_0}\int_\beta^{\gamma_1}\dots\int_\beta^{\gamma_{k-2}}\prod_{i=1}^{k-1}\eta_i w(\eta_i)d\eta_i\int_0^{\gamma_{k-1}} w(\eta_k)f(t-t_2\varsigma_{k-1})G^*_k(\eta_0, \eta_1, \dots, \eta_k)d\eta_k$$

The functions $P_\infty(t)$, $\tilde{P}_{0\pm}(t)$, $\tilde{P}_{1\pm}(t)$ are found from formulae (2.17), (2.18) and (4.14); the values of γ_i are found from the last two formulae of (4.20).

The integrodifferential equation of the punch motion on the elastic half-plane (2.14) in the time interval $nt_1 < t < (n+1)t_1$ considered is determined by its right-hand side, where instead of $Q(t)$ it is necessary to substitute $-P(t)$ as given by formula (4.21).

The condition for the punch to separate from the medium in the time interval (4.18), assuming that the punch has not separated from the half-plane when $0 < t \le nt_1$ (the time when separation begins $t = t_*$ (a = const)) is found from this condition), is given by the formula

$$\sum_{k=0}^{n} H((k+1)t_1 - t)C_k(t, a) = 0, \quad nt_1 < t < (n+1)t_1 \tag{4.22}$$

where

$$C_k(t, a) = \frac{(-1)^{k+1}H(t-kt_1)}{2\pi^{2k+1}K_-(0)}\sqrt{\frac{a}{c_2}}\int_0^{t-kt_1} f(\tau)R_k^*\left(\frac{t-\tau}{t_2}\right)d\tau, \quad k = 2, 3, \ldots, n$$

$$R_k^*(t) = \int_\beta^{t-(k-1)\beta} w(\eta_0)F_k^*(\eta_0, t)d\eta_0, \quad F_k^*(\eta_0, t) = \lim_{u\to 0} F_k(\eta_0, t, u) \tag{4.23}$$

$$F_k^*(\eta_0, t) = \int_\beta^{\gamma_0}\int_\beta^{\gamma_1}\ldots\int_\beta^{\gamma_{k-2}}\prod_{i=1}^{k-1}\eta_i w(\eta_i)du_i\int_0^{\gamma_{k-1}}\frac{\eta_{k-1}w(\eta_{k-1})}{\sqrt{t-\varsigma_{k-1}}}G_k^{**}(\eta_0, \eta_1, \ldots, \eta_{k-1}, t)d\eta_{k-1}$$

$$G_k^{**}(\eta_0, \eta_1, \ldots, \eta_{k-1}, t) = \int_\beta^\infty \frac{r(u_k)G_{k-1}^*(\eta_0, \eta_1, \ldots, \eta_{k-1})}{\eta_{k-1}+u_k}du_k$$

The functions $C_0(t, a)$ and $C_1(t, a)$ are found from formulae (3.2) and (4.17).

When $t > t_*$ ($nt_1 < t_* < (n+1)t_1$), condition (4.22) becomes an equation for determining the half-width of the contact area $a(t)$ at each step of the solution of integrodifferential Eq. (2.14) with $Q(t) = -P(t)$ from relations (4.21) from the definition of the law of the punch penetration $\varepsilon(t)$ into the elastic half-plane, and which occurs, together with the function $f(t)$, in the integrand of (4.23). It should be noted that when $t \ge t_*$ the solution $\varphi(x, t)$ of problem (4.19), obtained for the time interval (4.18), belongs to the class of functions that are continuous in the section $|x| \le 1$, as a result of satisfaction of condition (4.22), whereas for $nt_1 < t < t_*$ the solution $\varphi(x, t)$ belongs to the class of integrable functions with a root singularity when $x = \pm 1$.

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